

Outline

CISC 1100/1400 Structures of Comp. Sci./Discrete Structures

Chapter 1 Sets

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- ▶ Basic definitions
- ▶ Naming and describing sets
- ▶ Comparison relations on sets
- ▶ Set operations
- ▶ Principle of Inclusion/Exclusion

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Sets

- ▶ **Set:** a collection of objects (the *members* or *elements* of the set)
- ▶ **Set-lister notation:** curly braces around a list of the elements
 - ▶ $\{a, b, c, d, e, f\}$
 - ▶ $\{\text{Arizona, California, Massachusetts, 42, 47}\}$

- ▶ A set may contain other sets as elements:

$$\{1, 2, \{1, 2\}\}$$

- ▶ The empty set $\emptyset = \{\}$ contains no elements
- ▶ Can use variables (usually upper case letters) to denote sets

$$C = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$$

- ▶ **Universal set** (generally denoted U): contains all elements we might ever consider (only consider what matters)

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Enumerating the elements of a set

- ▶ Order doesn't matter

$$\{1, 2, 3\} = \{3, 1, 2\}$$

- ▶ Repetitions don't count

$$\{a, b, b\} = \{a, b\}$$

(better yet: don't repeat items in a listing of elements)

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Element notation

If A is a set, then

- ▶ $x \in A$ means “ x is an element of A ”
- ▶ $x \notin A$ means “ x is not an element of A ”

So

- ▶ $e \in \{a, e, i, o, u\}$
- ▶ $f \notin \{a, e, i, o, u\}$

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Some well-known sets

- ▶ Pretty much standard notations:
 - ▶ $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$: the set of *natural numbers* (non-negative integers).
 - ▶ $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, \dots\}$: the set of all *integers*.
 - ▶ \mathbb{Q} : the set of all *rational numbers* (fractions).
 - ▶ \mathbb{R} : the set of all *real numbers*.
- ▶ Less standard (but useful) notations:
 - ▶ \mathbb{Z}^+ is the set of positive integers.
 - ▶ \mathbb{Z}^- is the set of negative integers.
 - ▶ $\mathbb{Z}^{\geq 0}$ is the same as \mathbb{N} .
 - ▶ $\mathbb{R}^{>7}$ is the set of all real numbers greater than seven.

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Set builder notation

Rather than listing all the elements

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

(inconvenient or essentially impossible), describe sets via a *property*

$$A = \{x : p(x) \text{ is true}\}$$

Examples:

$$\mathbb{N} = \{x \mid x \in \mathbb{Z} \text{ and } x \geq 0\}$$

$$\mathbb{N} = \{x : x \in \mathbb{Z} \text{ and } x \geq 0\}$$

$$\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$$

$$\mathbb{N} = \{x \in \mathbb{Z} : x \geq 0\}$$

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Set builder notation (cont'd)

More examples:

$$\{x \in \mathbb{Q} : 2x = 7\} = \{3.5\}$$

$$\{x \in \mathbb{Z} : 2x = 7\} = \emptyset$$

$$\{2x \mid x \in \mathbb{Z}\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

$$\{x \in \mathbb{N} : \frac{1}{3}x \in \mathbb{Z}\} = \{0, 3, 6, 9, \dots\}$$

$$\{x \in \mathbb{R}^{\geq 0} : x^2 = 2\} = \{\sqrt{2}\}$$

$$\{x \in \mathbb{Q}^{\geq 0} : x^2 = 2\} = \emptyset$$

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Comparison relations on sets

- ▶ A is a *subset* of B (written " $A \subseteq B$ ") if each element of A is also an element of B .
 - ▶ $\{1, 3, 5\} \subseteq \{1, 2, 3, 4, 5\}$
 - ▶ $\{1, 2, 3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$
 - ▶ $\{1, 3, 6\} \not\subseteq \{1, 2, 3, 4, 5\}$
 - ▶ $A \subseteq A$ for any set A
 - ▶ $\emptyset \subseteq A$ for any set A
- ▶ A is a *proper subset* of B (written " $A \subset B$ ") if $A \subseteq B$ and $A \neq B$.
 - ▶ $\{1, 3, 5\} \subset \{1, 2, 3, 4, 5\}$
 - ▶ $\{1, 2, 3, 4, 5\} \not\subset \{1, 2, 3, 4, 5\}$
- ▶ \subset vs. \subseteq is somewhat like $<$ vs. \leq

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Element vs. subset

- ▶ \in means "is an element of"
- ▶ \subseteq means "is a subset of"

$A \subseteq B$ means if $x \in A$ then $x \in B$

- ▶ **Examples:** Let

$A = \{\text{purple, blue, orange, red}\}$ and $B = \{\text{blue}\}$.

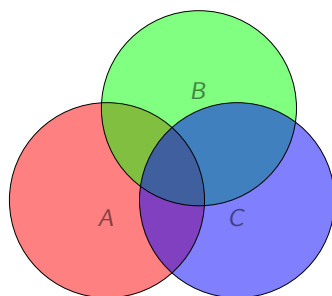
Fill in the missing symbol from the set $\{\in, \notin, \subseteq, \subset, \not\subseteq, =, \neq\}$ to correctly complete each of the following statements:

B	___	A	$\in, \subseteq, \subset, \neq$
blue	___	A	$\in, \not\subseteq, \neq$
green	___	A	$\in, \not\subseteq, \neq$
A	___	A	$\in, \subseteq, =$
$\{\text{blue, purple}\}$	___	B	$\in, \not\subseteq, \neq$
A	___	B	$\in, \not\subseteq, \neq$

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Venn Diagram

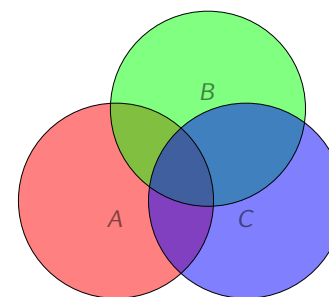
Diagram for visualizing sets and set operations



When necessary, indicate universal set via rectangle surrounding the set circles.

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Venn Diagram (cont'd)



For example, might have

$A = \{\text{Fordham students who've taken CISC 1100}\}$
 $B = \{\text{Fordham students who've taken CISC 1600}\}$
 $C = \{\text{Fordham students who've taken ECON 1100}\}$

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Set operations: Cardinality

The number of elements in a set is called its *cardinality*.

We denote the cardinality of S by $|S|$.

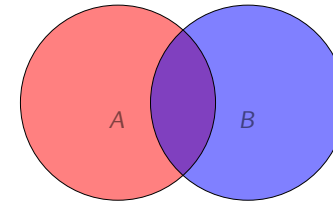
- ▶ Let $A = \{a, b, c, d, e, z\}$. Then $|A| = 6$.
- ▶ $|\{a, e, i, o, u\}| = 5$.
- ▶ $|\emptyset| = 0$.
- ▶ $|\{a, \{b, c\}, d, \{e, f, g\}, h\}| = 5$.

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Set operations: Union

Set of all elements belonging to *either* of two given sets:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



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Set operations: Union (examples)

Let

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{0, 2, 4, 6, 8\}$$

$$C = \{0, 5, 10, 15\}$$

Then

$$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8\}$$

$$B \cup A = \{0, 1, 2, 3, 4, 5, 6, 8\}$$

$$B \cup C = \{0, 2, 4, 5, 6, 8, 10, 15\}$$

$$(A \cup B) \cup C = \{0, 1, 2, 3, 4, 5, 6, 8, 10, 15\}$$

Let

$$L = \{e, g, b, d, f\}$$

$$S = \{f, a, c, e\}$$

Then

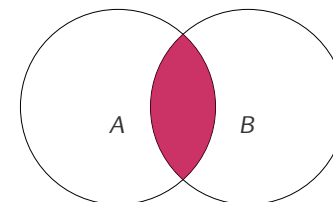
$$L \cup S = \{a, b, c, d, e, f, g\}$$

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Set operations: Intersection

Set of all elements belonging to *both* of two given sets:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



Note: We say that two sets are *disjoint* if their intersection is empty.

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Set operations: Intersection (examples)

Let

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{0, 2, 4, 6, 8\}$$

$$C = \{0, 5, 10, 15\}$$

Then

$$A \cap B = \{2, 4\}$$

$$B \cap A = \{2, 4\}$$

$$B \cap C = \{0\}$$

$$(A \cap B) \cap C = \emptyset$$

Let

$$L = \{e, g, b, d, f\}$$

$$S = \{f, a, c, e\}$$

Then

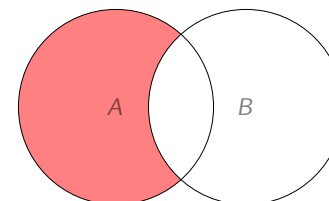
$$L \cap S = \{e, f\}$$

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Set operations: Difference

Set of all elements belonging to one set, but not another:

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$



Note that

$$|A - B| = |A| - |A \cap B|$$

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Set operations: Difference (examples)

Let

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{0, 2, 4, 6, 8\}$$

$$C = \{0, 5, 10, 15\}$$

Then

$$A - B = \{1, 3, 5\}$$

$$B - A = \{0, 6, 8\}$$

$$B - C = \{2, 4, 6, 8\}$$

$$C - B = \{5, 10, 15\}$$

$$(A - B) \cap (B - A) = \emptyset \quad (\text{Are you surprised by this?})$$

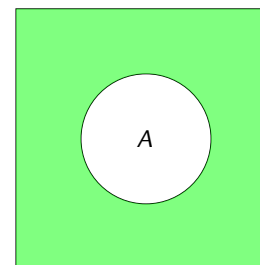
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Set operations: Complement

Set of all elements (of the universal set) that do *not* belong to a given set:

$$A' = U - A.$$

Venn diagrams dealing with complements generally use a surrounding rectangle to indicate the universal set U :



If U and A are finite sets, then

$$|A'| = |U - A| = |U| - |A|.$$

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Set operations: Complement (examples)

► Let

$$U = \{\text{red, orange, yellow, green, blue, indigo, violet}\}$$

$$P = \{\text{red, green, blue}\}$$

Then

$$P' = \{\text{orange, yellow, indigo, violet}\}$$

► Let E and O respectively denote the sets of even and odd integers. Suppose that our universal set is \mathbb{Z} . Then

$$E' = O$$

$$O' = E$$

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Set operations: Power Set

Set of all subsets of a given set

$$B \in \mathcal{P}(A) \text{ if and only if } B \subseteq A$$

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

$$\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$$

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

$$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

How many elements does $\mathcal{P}(A)$ have?

$$|\mathcal{P}(A)| = 2^{|A|},$$

i.e.,

$$\text{if } |A| = n, \text{ then } |\mathcal{P}(A)| = 2^n.$$

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Some basic laws of set theory

Here, U is a universal set, with $A, B, C, S \subseteq U$.

Name	Law
Identity	$S \cap U = S$
Identity	$S \cup \emptyset = S$
Complement	$S \cap S' = \emptyset$
Complement	$S \cup S' = U$
Double Complement	$(S')' = S$
Idempotent	$S \cap S = S$
Idempotent	$S \cup S = S$
Commutative	$A \cap B = B \cap A$
Commutative	$A \cup B = B \cup A$

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Some basic laws of set theory (cont'd)

Once again, U is a universal set, with $A, B, C, S \subseteq U$.

Name	Law
Associative	$(A \cap B) \cap C = A \cap (B \cap C)$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgan	$(A \cap B)' = A' \cup B'$
DeMorgan	$(A \cup B)' = A' \cap B'$
Equality	$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
Transitive	if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

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Set operations: Cartesian Product

- ▶ *Ordered pair*: Pair of items, in which order matters.
 - ▶ (1,2)... not the same thing as (2,1)
 - ▶ (red,blue)
 - ▶ (1,green)
- ▶ *Cartesian product* (also known as *set product*): Set of all ordered pairs from two given sets, i.e.,

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

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Set operations: Cartesian Product (examples)

Let

$$A = \{1, 2, 3\}$$

$$B = \{a, b, c\}$$

$$C = \{-1, 5\},$$

Then ...

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$$

$$C \times A = \{(-1, 1), (-1, 2), (-1, 3), (5, 1), (5, 2), (5, 3)\}$$

$$B \times C = \{(a, -1), (a, 5), (b, -1), (b, 5), (c, -1), (c, 5)\}$$

Note the following:

- ▶ $A \times B \neq B \times A$ (unless $A = B$)
- ▶ $|A \times B| = |A| \cdot |B|$ (that's why it's called "product").

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Principle of Inclusion/Exclusion

Suppose you run a fast-food restaurant. After doing a survey, you find that

- ▶ 25 people like ketchup on their hamburgers,
- ▶ 35 people like pickles on their hamburgers,
- ▶ 15 people like both ketchup *and* pickles on their hamburgers.

How many people like either ketchup *or* pickles (maybe both) on their hamburgers?

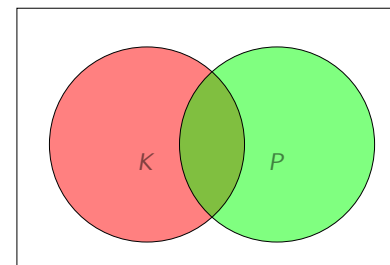
Let $K = \{\text{people who like ketchup}\}$ and

$P = \{\text{people who like pickles}\}$. Then

$$|K| = 25 \quad |P| = 35 \quad |K \cap P| = 15.$$

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Principle of Inclusion/Exclusion

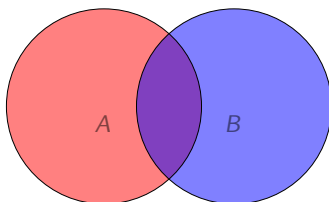


Since we don't want to count $K \cap P$ twice, we have

$$|K \cup P| = |K| + |P| - |K \cap P| = 25 + 35 - 15 = 45.$$

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Set operations: cardinalities of union and intersection



- ▶ Inclusion/exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- ▶ If A and B are disjoint, then

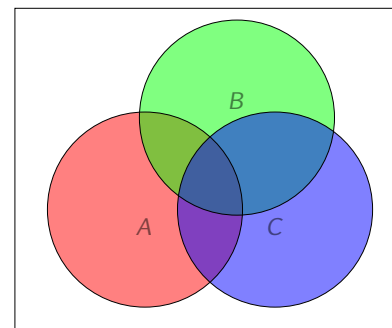
$$|A \cup B| = |A| + |B|$$

- ▶ See the example that animates this concept.
- ▶ What about three sets (hamburger eaters who like ketchup, pickles, and tomatoes)?

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Principle of Inclusion/Exclusion

For three sets:



$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

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Principle of Inclusion/Exclusion

Let K, P, T represent the sets of people who like ketchup, pickles, and tomatoes on their hamburgers. Suppose that

$$\begin{aligned} |K| &= 20 & |P| &= 30 & |T| &= 45 \\ |K \cap P| &= 10 & |K \cap T| &= 12 & |P \cap T| &= 13 \\ |K \cap P \cap T| &= 8. \end{aligned}$$

Then

$$\begin{aligned} |K \cup P \cup T| &= |K| + |P| + |T| - \\ &\quad |K \cap P| - |K \cap T| - |P \cap T| + \\ &\quad |K \cap P \cap T| \\ &= 20 + 30 + 45 - 10 - 12 - 13 + 8 \\ &= 68 \end{aligned}$$

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