

CISC 1100/1400
Structures of Comp. Sci./Discrete Structures
Chapter 3
Logic

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Logical (or illogical?) reasoning

- ▶ Is this a valid argument?
 - ▶ All men are mortal.
 - ▶ Socrates is a man.
 - ▶ Therefore, Socrates is mortal.

Valid or not? Yes!

- ▶ Is this a valid argument?
 - ▶ If we finish our homework, then we will go out for ice cream.
 - ▶ We are going out for ice cream.
 - ▶ Therefore we finished our homework.

Valid or not? No!

- ▶ How to recognize the difference?

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Outline

- ▶ Propositional logic
 - ▶ Logical operations
 - ▶ Propositional forms
 - ▶ From English to propositions
 - ▶ Propositional equivalence
- ▶ Predicate logic
 - ▶ Quantifiers
 - ▶ Some rules for using predicates

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Propositional logic

Proposition: A statement that is either true or false:

- ▶ $2 + 2 = 4$.
- ▶ $2 + 2 = 5$.
- ▶ It rained yesterday in Manhattan.
- ▶ It will rain tomorrow in Manhattan.

These are *not* propositions:

- ▶ $x + 2 = 4$.
- ▶ Will it rain today in Manhattan?
- ▶ Colorless green ideas sleep furiously.

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Propositional logic (cont'd)

- ▶ *Truth value* of a proposition (T, F)
- ▶ Propositional variables: lower case letters (p, q, \dots) (Analogous to variables in algebra.)
 - ▶ p = "A New York City subway fare is \$2.50."
 - ▶ q = "It will rain today in Manhattan."
 - ▶ r = "All multiples of four are even numbers."

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Logical operations: negation

- ▶ *Negation*, the NOT operation: reverses a truth value.
- ▶ Negation is a *unary operation*: only depends on one variable.
- ▶ Negation of p is denoted p' . (Some books use other notations, such as \bar{p} , $\sim p$, or $\neg p$.)
- ▶ Can display via a *truth table*

p	p'
T	F
F	T

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Logical operations: conjunction, disjunction

- ▶ The remaining operations we discuss are *binary operations*: they depend on two variables (also called *connectives*).
- ▶ *Conjunction*, the AND operation: true if *both* operands are true. Denote by \wedge .
- ▶ *Disjunction*, the (inclusive) OR operation: true if *either* operand is true (including both). Denote by \vee .
- ▶ Truth tables:


p	q	$p \wedge q$	p	q	$p \vee q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	F	T	T
F	F	F	F	F	F

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Logical operations: exclusive or

- ▶ The inclusive or \vee is not the "or" of common language.
- ▶ That role is played by *exclusive or* (XOR), denoted \oplus .
- ▶ Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- ▶ Be careful to distinguish between or and xor! 

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Logical operations: conditional

- ▶ Denoted $p \Rightarrow q$.
- ▶ Captures the meaning of
 - ▶ If p , then q .
 - ▶ p implies q .
 - ▶ p only if q .
 - ▶ p is sufficient for q .
 - ▶ q is necessary for p .
- ▶ Truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- ▶ First two rows are “obvious”.
- ▶ Last two rows are not so obvious:
“One can derive anything from a false hypothesis.”

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Logical operations: biconditional

- ▶ Denoted $p \Leftrightarrow q$
- ▶ Captures the meaning of
 - ▶ p if and only if q .
 - ▶ p is necessary and sufficient for q .
 - ▶ p is logically equivalent to q .
- ▶ Truth table:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

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Propositional Forms

In arithmetic and algebra, you learned how to build up complicated arithmetic expressions, such as

- ▶ $1 + 2$
- ▶ $-(1 + 2)$
- ▶ 3×4
- ▶ $-(1 + 2)/(3 \times 4)$
- ▶ $-(1 + 2)/(3 \times 4) + (5 + 6 \times 7)/(8 + 9) - 10$

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Propositional Forms (cont'd)

- ▶ Use connectives to build complicated expressions from simpler ones, *or*
- ▶ break down complicated expressions as being simpler subexpressions, connected by connectives.
- ▶ **Example:** $-(1 + 2)/(3 \times 4) + (5 + 6 \times 7)/(8 + 9) - 10$ consists of

$$-(1 + 2)/(3 \times 4) \quad \text{and} \quad (5 + 6 \times 7)/(8 + 9) - 10,$$

connected by +.

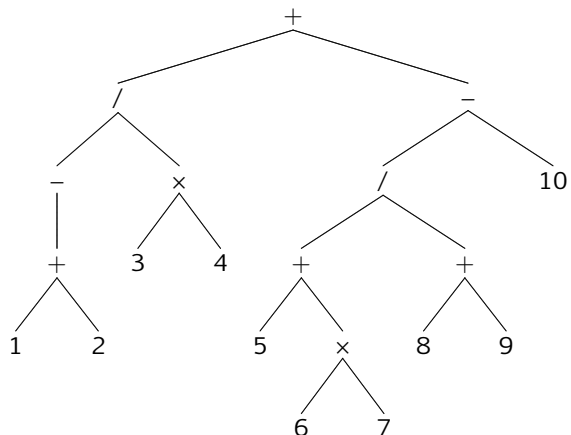
- ▶ Now break down these two subexpressions.
- ▶ Now break down the four sub-subexpressions.
- ▶ And so forth.

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Propositional Forms (cont'd)

Systematize the process via a *parse tree*.

Parse tree for $-(1 + 2)/(3 \times 4) + (5 + 6 \times 7)/(8 + 9) - 10$:



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Propositional Forms (cont'd)

We're inherently using the following rules:

1. Parenthesized subexpressions are evaluated first.
2. Operations have a *precedence hierarchy*:
 - 2.1 Unary operations (for example, -1) are done first.
 - 2.2 Multiplicative operations (\times and $/$) are done next.
 - 2.3 Additive operations ($+$ and $-$) are done last.
3. In case of a tie (two additive operations or two multiplicative operations), the remaining operations are done from left to right.

These guarantee that (e.g.) $2 + 3 \times 4$ is 14, rather than 20.

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Propositional Forms (cont'd)

- ▶ Now jump from numerical algebra to propositional algebra.
- ▶ We can build new (complicated) propositions out of old (simpler) ones.
- ▶ **Example:** $[(p \vee q) \wedge ((p') \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee (p \wedge r)]$ consists of

$$(p \vee q) \wedge ((p') \vee r) \quad \text{and} \quad (p \Leftrightarrow q) \vee (p \wedge r),$$

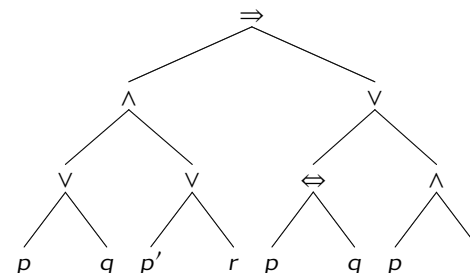
connected by \Rightarrow .

- ▶ Now break down these two subexpressions.
- ▶ Now break down the four sub-subexpressions.
- ▶ And so forth.

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Propositional Forms (cont'd)

Parse tree for $[(p \vee q) \wedge ((p') \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee (p \wedge r)]$:



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Propositional Forms (cont'd)

- ▶ The expression

$$[(p \vee q) \wedge ((p') \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee (p \wedge r)]$$

is completely parenthesized (and hard to read).

- ▶ If we agree upon (standard) precedence rules, can get rid of extraneous parentheses.
 1. Parenthesized subexpressions are evaluated first.
 2. Operations have a *precedence hierarchy*:
 - 2.1 Unary negations ($'$) are done first.
 - 2.2 Multiplicative operations (\wedge) are done next.
 - 2.3 Additive operations (\vee, \oplus) are done next.
 - 2.4 The conditional-type operations (\Rightarrow and \Leftrightarrow) are done last.
 3. In case of a tie (two operations at the same level in the hierarchy), operations are done in a left-to-right order, *except* for the conditional operator \Rightarrow , which is done in a right-to-left order. That is, $p \Rightarrow q \Rightarrow r$ is interpreted as $p \Rightarrow (q \Rightarrow r)$.

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Propositional Forms (cont'd)

So can replace

$$[(p \vee q) \wedge ((p') \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee (p \wedge r)]$$

by

$$[(p \vee q) \wedge (p' \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee p \wedge r]$$

or even

$$(p \vee q) \wedge (p' \vee r) \Rightarrow (p \Leftrightarrow q) \vee p \wedge r.$$

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Propositional Forms (cont'd)

- ▶ Precedence rules are too hard to remember!
- ▶ Let's simplify!
 1. Parenthesized subexpressions come first.
 2. Next comes the only unary operation ($'$).
 3. Next comes the only multiplicative operation (\wedge).
 4. Next comes the additive operations (\vee, \oplus).
 5. Use parentheses if you have *any* doubt.
Always use parentheses if you have multiple conditionals.
 6. Evaluate ties left-to-right.

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From English to Propositions

- ▶ Can use propositional forms to capture logical arguments in English.
- ▶ Help to expose logical fallacies.
- ▶ **Example:** Alice will have coffee or Bob will go to the beach. Let

$a = \text{"Alice will have coffee"}$

$b = \text{"Bob will go to the beach"}$

Solution? $a \vee b$.

- ▶ **Example:** If I make peanut butter sandwiches for lunch, then Carol will be disappointed. Let

$p = \text{"I will make peanut butter sandwiches"}$

$c = \text{"Carol will be disappointed"}$

Solution? $p \Rightarrow c$.

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From English to Propositions (cont'd)

- ▶ **Example:** If Alice will have coffee and Bob will go to the beach, then either Carol will be disappointed or I will make peanut butter sandwiches.

Solution? $a \wedge b \Rightarrow c \vee p$

- ▶ **Example:**

Alice will have coffee and
Bob will not go to the beach

if and only if

Carol will be disappointed and
I will not make peanut butter sandwiches.

Solution? $(a \wedge b') \Leftrightarrow (c \wedge p')$

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Propositional Equivalence

High school algebra: establishes many useful rules, such as

$$\begin{aligned}a + b &= b + a, \\ a \times (b + c) &= a \times b + a \times c, \\ -(a + b) &= (-a) + (-b),\end{aligned}$$

Anything analogous for propositions?

- ▶ How to state them? (No equal sign.)
- ▶ How to prove correct rules?
- ▶ How to disprove incorrect “rules”?

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Propositional Equivalence (cont'd)

- ▶ **Logical equivalence:** $p \equiv q$ means
 p is true if and only if q is true

- ▶ Beware! 

- ▶ $p \equiv q$ is *not* a proposition; it's a statement *about* propositions.
- ▶ $p \equiv q$ is a statement in a *metalanguage* about propositions.
- ▶ \equiv is a *metasymbol* in this language.

- ▶ Analogous to

$$\begin{aligned}a + b &= b + a, \\ a \times (b + c) &= a \times b + a \times c, \\ -(a + b) &= (-a) + (-b),\end{aligned}$$

we might *conjecture* that

$$\begin{aligned}p \vee q &\equiv q \vee p, \\ p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r), \\ (p \vee q)' &\equiv p' \vee q'.\end{aligned}$$

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Propositional Equivalence (cont'd)

- ▶ Want to prove (or disprove) conjectured identities such as

$$\begin{aligned}p \vee q &\equiv q \vee p, \\ p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r), \\ (p \vee q)' &\equiv p' \vee q'.\end{aligned}$$

- ▶ How? Use a truth table.
- ▶ Suppose that p and q are propositional formulas.
The equivalence $p \equiv q$ is true iff the truth tables for p and q are identical.

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Propositional Equivalence (cont'd)

Example: Is it true that $p \vee q \equiv q \vee p$?

p	q	$p \vee q$	p	q	$q \vee p$
T	T	T	T	T	T
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	F	F	F

They match! So $p \vee q \equiv q \vee p$.

More compact form:

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

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Propositional Equivalence (cont'd)

Example: Is it true that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$?

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

So $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

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Propositional Equivalence (cont'd)

► How to organize the table?

- Two variables: TT, TF, FT, FF
- Three variables: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF.
- General pattern?
 - Rightmost variable alternates: TFTFTFTF ...
 - Next alternates in pairs: TTFFTTFF ...
 - Next alternates in quadruples: TTTTFFFTTTTFFFF ...

► Size of table?

- Two variables? 4 rows.
- Three variables? 8 rows.
- n variables? 2^n rows.
- Since $2^{10} = 1024$, you don't want to do a 10-variable table.

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Propositional Equivalence (cont'd)

Example: Is it true that $(p \vee q)' \equiv p' \vee q'$?

p	q	$p \vee q$	$(p \vee q)'$	p'	q'	$p' \vee q'$
T	T	T	F	F	F	F
T	F	T	F	F	T	T
F	T	T	F	T	F	T
F	F	F	T	T	T	T

So it is *not* true that $(p \vee q)' \equiv p' \vee q'$!

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Propositional Equivalence (cont'd)

Example: Rather than $(p \vee q)' \equiv p' \vee q'$, the correct formula is $(p \vee q)' \equiv p' \wedge q'$

p	q	$p \vee q$	$(p \vee q)'$	p'	q'	$p' \wedge q'$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The formula $(p \wedge q)' \equiv p' \vee q'$ is also correct.

These formulas

$$(p \vee q)' \equiv p' \wedge q'$$

$$(p \wedge q)' \equiv p' \vee q'$$

are called *deMorgan's laws*.

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Propositional Equivalence (cont'd)

Some well-known propositional laws (we haven't proved them all):

Double Negation	$(p')' \equiv p$
Idempotent	$p \wedge p \equiv p$
Idempotent	$p \vee p \equiv p$
Commutative	$p \wedge q \equiv q \wedge p$
Commutative	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Associative	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
DeMorgan	$(p \wedge q)' \equiv (p') \vee (q')$
DeMorgan	$(p \vee q)' \equiv (p') \wedge (q')$
Modus Ponens	$[(p \Rightarrow q) \wedge p] \Rightarrow q$
Modus Tollens	$[(p \Rightarrow q) \wedge q'] \Rightarrow p'$
Contrapositive	$(p \Rightarrow q) \equiv (q' \Rightarrow p')$
Implication	$(p \Rightarrow q) \equiv (p' \vee q)$

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Propositional Equivalence (cont'd)

The preceding table is similar to the table of set identities from Chapter 1, e.g., we have

$$(p \wedge q)' \equiv p' \vee q' \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

It turns out that we can use a propositional law to easily prove the analogous set identity.

Example: Show that $(A \cap B)' = A' \cup B'$.

Solution: Must show that any element of $(A \cap B)'$ is an element of $A' \cup B'$, and vice versa. But

$$\begin{aligned} x \in (A \cap B)' &\iff (x \in A \cap B)' \iff (x \in A \wedge x \in B)' \\ &\iff (x \in A)' \vee (x \in B)' \\ &\iff (x \in A') \vee (x \in B') \\ &\iff x \in A' \cup B', \end{aligned}$$

as required. \square

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Propositional Equivalence (cont'd)

Once we've proved a given propositional law, we can use it to help prove new ones.

Example: Let's prove the *exportation identity*

$$[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)].$$

We have

$$\begin{aligned} (p \wedge q) \Rightarrow r &\equiv (p \wedge q)' \vee r && \text{implication} \\ &\equiv (p' \vee q') \vee r && \text{DeMorgan} \\ &\equiv p' \vee (q' \vee r) && \text{associative} \\ &\equiv p' \vee (q \Rightarrow r) && \text{implication} \\ &\equiv p \Rightarrow (q \Rightarrow r) && \text{implication} \end{aligned}$$

as required. \square

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Propositional Equivalence (cont'd)

► **Duality:** If p is a proposition that only uses the operations \neg , \wedge , and \vee . If we replace all instances of \wedge , \vee , \neg , and \neg in p by \vee , \wedge , \neg , and \neg , respectively, we get a new proposition p^* , which is called the *dual* of p .

► **Example:** The duals of

$$p \wedge (q \vee r) \quad \text{and} \quad (p \wedge q) \vee (p \wedge r)$$

are

$$p \vee (q \wedge r) \quad \text{and} \quad (p \vee q) \wedge (p \vee r).$$

► **Duality Principle:** If two propositions (which only use the operations \neg , \wedge , and \vee) are equivalent, then their duals are equivalent. (Be lazy—save half the work!)

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Propositional Equivalence (cont'd)

Example: Since the duals

$$p \wedge (q \vee r) \quad \text{and} \quad (p \wedge q) \vee (p \wedge r)$$

are

$$p \vee (q \wedge r) \quad \text{and} \quad (p \vee q) \wedge (p \vee r)$$

and we had earlier proved that

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r),$$

we now know that

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

“for free”.

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Propositional Equivalence (cont'd)

Example: Since the duals of

$$(p \vee q)' \quad \text{and} \quad p' \wedge q'$$

are

$$(p \wedge q)' \quad \text{and} \quad p' \vee q'$$

and we had earlier proved that

$$(p \vee q)' \equiv p' \wedge q',$$

we now know that

$$(p \wedge q)' \equiv p' \vee q'$$

“for free”.

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Indirect Proofs

Sometimes you can prove a statement with a direct approach.

Example: Show that the square of an odd number is also an odd number.

Solution: Let m be an odd number; want to show that m^2 is odd.

► Write $m = 2n + 1$ for $n \in \mathbb{Z}$.

► Then

$$m^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1.$$

Let $k = 2n^2 + 2n \in \mathbb{Z}$. Then $m^2 = 2k + 1$, and so m^2 is odd. \square

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Indirect Proofs (cont'd)

Sometimes a “frontal attack” doesn’t work. So we use an “sneak attack”, more properly called an *indirect proof*.

Two such techniques:

- ▶ *Proof by contradiction*. Show that if the statement to be proved is false, then a contradiction results.
- ▶ *Proving the contrapositive*. Rather than directly proving an implication $p \Rightarrow q$, prove its contrapositive $q' \Rightarrow p'$.

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Indirect Proofs (cont'd)

Example: Show that if the square of an integer is even, then that integer is even.

Solution: Let $m \in \mathbb{Z}$. We want to show that

$$m^2 \text{ is even} \Rightarrow m \text{ is even.}$$

We can do this by establishing its contrapositive. But the contrapositive is

$$m \text{ is odd} \Rightarrow m^2 \text{ is odd,}$$

which we did previously. So we’re done!!

□

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Indirect Proofs (cont'd)

Example: Show that $\sqrt{2}$ is an irrational number.

Solution: Let’s do a proof by contradiction. Rather than showing $\sqrt{2} \notin \mathbb{Q}$, let’s assume that $\sqrt{2} \in \mathbb{Q}$, and show how this leads to a contradiction.

So write $\sqrt{2} = p/q$ for $p, q \in \mathbb{Z}^+$, where $q \neq 0$ and where p and q have no common factor other than 1 (i.e., the fraction p/q is “reduced to lowest terms”). Then

$$\sqrt{2} = \frac{p}{q} \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2 \text{ is even}$$

$\Rightarrow p$ is even (see previous slide)

$\Rightarrow p = 2r$ for some positive integer r

$\Rightarrow (2r)^2 = p^2 = 2q^2$ (Remember that $p^2 = 2q^2$!)

$\Rightarrow 4r^2 = 2q^2 \Rightarrow 2r^2 = q^2 \Rightarrow q^2$ is even

$\Rightarrow q$ is even (again using previous slide)

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Indirect Proofs (cont'd)

Example (cont'd): Show that $\sqrt{2}$ is an irrational number.

Solution (cont'd): We’re doing a proof by contradiction. Rather than showing $\sqrt{2} \notin \mathbb{Q}$, we are trying to show how the assumption $\sqrt{2} \in \mathbb{Q}$ leads to a contradiction.

We wrote $\sqrt{2} = p/q$ for $p, q \in \mathbb{Z}^+$, where $q \neq 0$ and where p and q have no common factor other than 1 (i.e., the fraction p/q is “reduced to lowest terms”).

Previous slide: Both p and q are even, i.e., they are both exact integer multiples of 2.

This contradicts the assumption that p, q have no common factor (other than 1)!

Hence we cannot write $\sqrt{2} = p/q$ for $p, q \in \mathbb{Z}^+$, where $q \neq 0$ and where p and q have no common factor other than 1.

Hence $\sqrt{2} \notin \mathbb{Q}$.

□

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An Example From Lewis Carroll

Given the following facts:

1. All babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.

Prove that babies cannot manage crocodiles.

Let b , c , d , and l denote the status of being a baby, being able to manage a crocodile, being despised, and being logical. Then

1. $b \Rightarrow l'$.
2. $c \Rightarrow d'$.
3. $l' \Rightarrow d$.

We now have

- ▶ $b \Rightarrow d$, using (1), (3), transitive law.
- ▶ $(d')' \Rightarrow c'$, using (2), contrapositive law.
- ▶ $d \Rightarrow c'$, since $(d')' \equiv d$ (double negation law).
- ▶ Hence transitive law gives $b \Rightarrow c'$. □

See text for a 10-fact example.

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Predicate Logic

Want to symbolically state the classical syllogism

- ▶ All men are mortal.
- ▶ Socrates is a man.
- ▶ Therefore, Socrates is mortal.

Let

$$\begin{aligned} \text{man}(x) &= \text{"x is a man"} \\ \text{mortal}(x) &= \text{"x is mortal"} \end{aligned}$$

We can agree that $\text{man}(\text{Socrates})$ is (was?) true and that

$$\text{man}(x) \Rightarrow \text{mortal}(x) \text{ for any person } x.$$

Our natural conclusion? $\text{mortal}(\text{Socrates})$ is true.

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Predicate Logic (cont'd)

- ▶ A *predicate* is a formula that contains a variable, that becomes a proposition when we substitute a particular value for the variable.
- ▶ In other words, plug in a value and get a truth value (T or F).
- ▶ Examples: $\text{man}(x)$ or $\text{mortal}(x)$.
- ▶ Can have more than one variable, e.g.,

$$\text{older}(x, y) = \text{"x is older than y"}.$$

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Predicate Logic (cont'd)

For example, suppose that $\text{four}(t)$ means that $t \in \mathbb{Z}$ is divisible by 4 (in other words, t is an exact multiple of 4). Then:

x	$\text{four}(x)$	truth value of $\text{four}(x)$
\vdots	\vdots	\vdots
-4	-4 is divisible by 4	T
-3	-3 is divisible by 4	F
-2	-2 is divisible by 4	F
-1	-1 is divisible by 4	F
0	0 is divisible by 4	T
1	1 is divisible by 4	F
2	2 is divisible by 4	F
3	3 is divisible by 4	F
4	4 is divisible by 4	T
\vdots	\vdots	\vdots

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Quantifiers

- ▶ How to transform a predicate $p(x)$ (where x varies over a set S) into a proposition?
- ▶ **Universal quantification:** We ask that $p(x)$ be true for *all* $x \in S$. We let


$$\forall x \in S, p(x)$$

denote the proposition “For all elements $x \in S$, $p(x)$ is true.”

- ▶ **Existential quantification:** We ask that $p(x)$ be true for *some* $x \in S$. We let

$$\exists x \in S: p(x)$$

denote the proposition “There exists some $x \in S$ such that $p(x)$ is true.”

- ▶ Note the slight punctuation difference (comma vs. colon). 

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Quantifiers (cont'd)

- ▶ Let

$\text{four}(x) =$ “ x is divisible by four.” for any $x \in \mathbb{Z}$.

- ▶ $\forall x \in \mathbb{Z}, \text{four}(x)$ is false.
- ▶ $\exists x \in \mathbb{Z}: \text{four}(x)$ is true.
- ▶ Consider the predicate $x > y$ over $x, y \in \mathbb{Z}$.
 - ▶ $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x > y$ is false
 - ▶ $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}: x > y$ is true

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Some Rules for Using Predicates

- ▶ **Classical syllogism:** Suppose that
 - ▶ $p(x)$ and $q(x)$ are predicates, with x varying over some set S .
 - ▶ $p(x) \Rightarrow q(x)$ for any $x \in S$.

Suppose further that $p(a)$ is true for some $a \in S$.

Then $q(a)$ is true.

We can write this symbolically as

$$[\forall x \in S, p(x) \Rightarrow q(x) \wedge a \in S \wedge p(a)] \Rightarrow q(a).$$

- ▶ **Negation laws:**

$$[\exists x \in S: p(x)]' \equiv [\forall x \in S, p'(x)]$$

and

$$[\forall x \in S, p(x)]' \equiv [\exists x \in S: p'(x)].$$

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Predicates Having More Than One Variable

- ▶ Any given variable might not be quantified.
- ▶ The quantified variables might be quantified differently.
- ▶ **Example:** Let P be a set of people, T be a set of temperatures. Define “ $\text{beach}(p, t)$ ” to mean that “person p will go to the beach if the temperature reaches t degrees”.

Quantification choices?

- ▶ No quantification. $\text{beach}(p, t)$ is a two-variable predicate.
- ▶ We can quantify in one variable.
Quantifying over p gives the following predicates in t :

$$\exists p \in P: \text{beach}(p, t)$$

$$\forall p \in P, \text{beach}(p, t).$$

Quantifying over t gives the following predicates in p :

$$\exists t \in T: \text{beach}(p, t)$$

$$\forall t \in T, \text{beach}(p, t).$$

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Predicates Having More Than One Variable (cont'd)

► Quantification example (cont'd)

- We can quantify in both variables, getting the propositions:

$$\exists p \in P: [\exists t \in T: \text{beach}(p, t)]$$

$$\exists p \in P: [\forall t \in T, \text{beach}(p, t)]$$

$$\forall p \in P, [\exists t \in T: \text{beach}(p, t)]$$

$$\forall p \in P, [\forall t \in T, \text{beach}(p, t)].$$

(Many people would omit the brackets.)

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Logic Circuits

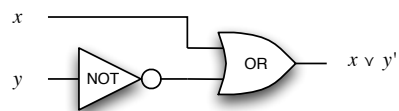
- Computers are built from a small list of basic components.
- Technology changes rapidly.
- So don't want to describe hardware in terms of the basic components.
- Instead, use *logic gates* to build *logic circuits*.
- Absence or presence of electrical signal signifies "false" or "true".
- Since $\{', \wedge, \vee\}$ is universal set of Boolean operations, it suffices to use logic gates for same:



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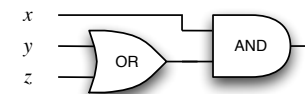
Example: Build a circuit that computes $x \vee y'$.

Solution:



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Example: Find the Boolean expression computed by



Solution: This circuit computes $x \wedge (y \vee z)$.

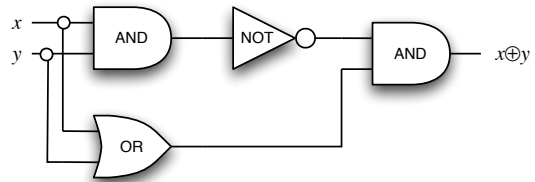
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Example: How would we build an XOR circuit?

Solution: From Exercise 3.3.12, we have

$$x \oplus y \equiv (x \wedge y)' \wedge (x \vee y).$$

So the desired logic circuit is given by



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Example: A building is protected by a fire alarm, a smoke alarm, and a burglar alarm. This alarm system is controlled by three inputs:

- ▶ The signal b coming from the burglar alarm.
- ▶ The signal f coming from the fire alarm.
- ▶ The signal s coming from the smoke alarm.

There will be two outputs:

- ▶ a signal telling us to call the fire department, which we'll denote by a , and
- ▶ a signal telling us to call the police department, which we'll denote by p .

Solution: Thinking about this a bit, it should be clear that

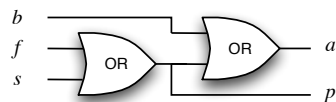
$$p = b \vee f \vee s \quad \text{and} \quad a = f \vee s.$$

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Example (cont'd): Since

$$p = b \vee f \vee s \quad \text{and} \quad a = f \vee s,$$

the logic circuit is given by



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