

Why talk about counting in a college-level course?

CISC 1100/1400 Structures of Comp. Sci./Discrete Structures

Chapter 6 Counting

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- ▶ Counting isn't as easy as it looks.
 - ▶ Simple sets: trivial to count.
 - ▶ Complicated sets: hard to count.
 - ▶ Facebook FOAF.
 - ▶ Number of ways to fill a committee.
 - ▶ Number of ways to fill a slate of officers.
 - ▶ Number of outcomes in a game (chess, poker, ...).
 - ▶ Methodically enumerating a set.
- ▶ Connection between counting and probability theory.

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Outline

- ▶ Counting and how to count
- ▶ Elementary rules for counting
 - ▶ The addition rule
 - ▶ The multiplication rule
 - ▶ Using the elementary rules for counting together
- ▶ Permutations and combinations
- ▶ Additional examples

Counting and how to count

- ▶ Some things are easy to count (e.g., number of students in this class).
- ▶ Some things are harder to count.
- ▶ **Example:** You are asked to select a pair of men's jeans.
 - ▶ Four styles are available (standard fit, loose fit, boot fit, and slim fit).
 - ▶ Each style comes in two colors (blue or black).
- ▶ You *could* list all possibilities for this problem.

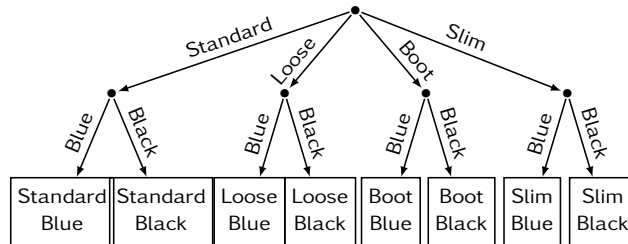
Color	Jeans Style			
	Standard	Loose	Boot	Slim
Blue	Standard-Blue	Loose-Blue	Boot-Blue	Slim-Blue
Black	Standard-Black	Loose-Black	Boot-Black	Slim-Black

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Counting and how to count (cont'd)

- ▶ This doesn't generalize.
 - ▶ What if more than two "features"?
- ▶ One idea: Use a *tree structure* to help you enumerate the choices.



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Counting and how to count (cont'd)

Example: We toss a penny, a nickel, and a dime into the air. How many different configurations?

- ▶ How to encode? As a triple:
(penny's state, nickel's state, dime's state)

- ▶ Configurations?

$$C = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}.$$

- ▶ How many configurations? 8.
- ▶ How to count configurations without listing?

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Elementary rules of counting

- ▶ Two basic rules:
 - ▶ Addition rule
 - ▶ Multiplication rule
- ▶ Using these rules together

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Elementary rules of counting: the addition rule

- ▶ **Example:** You need to purchase one shirt of any kind. The store has five short sleeve shirts and eight long sleeve shirts. How many possible ways are there to choose a shirt?
- ▶ **Solution:** $8 + 5 = 13$.
- ▶ **Addition rule:**
 - ▶ If we have two choices C_1 and C_2 , with C_1 having a set O_1 of possible outcomes and C_2 having a set O_2 of possible outcomes, with $|O_1| = n_1$ and $|O_2| = n_2$, then the total number of outcomes for C_1 or C_2 occurring is $n_1 + n_2$.
 - ▶ If we have k choices C_1, \dots, C_k having n_1, \dots, n_k possible outcomes, then the total number of ways of C_1 occurring or C_2 occurring or ... or C_k occurring is $n_1 + n_2 + \dots + n_k$.
- ▶ Fairly straightforward.

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Elementary rules of counting: the multiplication rule

- ▶ In our jeans example,
of jeans configurations =
(# number of styles) \times (# of colors)
- ▶ **Multiplication rule:**
 - ▶ If we have two choices C_1 and C_2 , with C_1 having a set O_1 of possible outcomes and C_2 having a set O_2 of possible outcomes, with $|O_1| = n_1$ and $|O_2| = n_2$, then the total number of possible outcomes for C_1 and C_2 occurring is $n_1 \times n_2$.
 - ▶ More generally, if we have k choices C_1, \dots, C_k having n_1, \dots, n_k possible outcomes, then the total number of ways of C_1 occurring and C_2 occurring and ... and C_k occurring is $n_1 \times n_2 \times \dots \times n_k$.
- ▶ Roughly speaking:
 - ▶ addition rule: "or" rule
 - ▶ multiplication rule: "and" rule

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Elementary rules of counting: the multiplication rule (cont'd)

- Example:** Solve jeans problem via multiplication rule ...
- ▶ four styles (standard, loose, slim, and boot fits) and
 - ▶ two colors (black, blue)

Solution: Our choices?

$C_1 =$ "choose the jeans style",

$C_2 =$ "choose the jeans color".

Our outcomes?

$O_1 =$ {standard fit, loose fit, boot fit, slim fit},

$O_2 =$ {black, blue}.

Now determine the cardinalities of the sets:

$$n_1 = |O_1| = 4 \quad n_2 = |O_2| = 2.$$

Now we apply the multiplication rule

$$\text{Total number of outcomes} = n_1 \times n_2 = 4 \times 2 = 8.$$

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Elementary rules of counting: the multiplication rule (cont'd)

- ▶ Why does the multiplication rule work?
- ▶ The set of possible outcomes is for O_1 and O_2 occurring is $O_1 \times O_2$.
- ▶ We know that $|O_1 \times O_2| = |O_1| \cdot |O_2|$.
- ▶ This is the multiplication rule!

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Elementary rules of counting: the multiplication rule (cont'd)

Example: Suppose that you flip a coin twice and record the outcome (head or tail) for each flip. How many possible outcomes are there?

Solution: There are two choices, C_1 and C_2 , corresponding to the two coin flips. C_1 and C_2 must occur, so the multiplication rule applies. Each choice has two possible outcomes, thus $n_1 = 2$ and $n_2 = 2$. Thus by the multiplication principle of counting, there are $2 \times 2 = 4$ possible outcomes.

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Elementary rules of counting: the multiplication rule (cont'd)

Example: You are asked to flip a coin five times and to record the outcome (head or tail) for each flip. How many possible outcomes are there?

Solution:

- ▶ This example differs from the previous one only in that there are five choices instead of two.
- ▶ For each choice there are two possible outcomes.
- ▶ The total number of outcomes is

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

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Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. If a number may be selected more than once, then how many ways can you fill out the lottery card?

Solution:

- ▶ There are five choices, corresponding to the five numbers that you must choose.
- ▶ Each of the five choices must occur, so the multiplication rule applies.
- ▶ Each choice has twenty possible outcomes (i.e., you pick a number between 1 and 20).
- ▶ There are

$$20 \times 20 \times 20 \times 20 \times 20 = 20^5 = 3,200,000$$

possible ways to fill out the lottery card.

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Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. The numbers are chosen by the lottery commission from a bin and once a number is chosen it is discarded and cannot be chosen again. In how many ways can you fill out the lottery card?

Elementary rules of counting: the multiplication rule (cont'd)

Solution:


- ▶ Close to the previous one, but a number cannot be chosen more than once.
- ▶ Hence, the number of possible outcomes for each choice is progressively reduced by one.
- ▶ Number the five choices $C_1 \dots C_5$ such that C_1 corresponds to the first number selected and C_5 to the last number selected.
- ▶ The number of outcomes for C_1 is 20, for C_2 is 19, for C_3 is 18, for C_4 is 17 and for C_5 is 16.
- ▶ Thus the number of possible outcomes is

$$20 \times 19 \times 18 \times 17 \times 16 = 1,860,480.$$

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Elementary rules of counting: the multiplication rule (cont'd)

- ▶  Don't be misled by the word "and"!
- ▶ **Example:** How many ways are there to choose one class among 5 day classes and 2 evening classes?
- ▶ **Solution:** $5 + 2 = 7$ ways.

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Elementary rules of counting: combining the rules together

- ▶ **Example:** How many odd three-digit numbers are there (allowing leading zeros, such as 007)?
- ▶ **First solution:**
 - ▶ We have three choices, one per digit. Let C_1, C_2, C_3 denote the choices for the first, second, third digits.
 - ▶ $O_1 = O_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, while $O_3 = \{1, 3, 5, 7, 9\}$.
 - ▶ So $|O_1| = 10, |O_2| = 10, |O_3| = 5$.
 - ▶ Hence there are $10 \times 10 \times 5 = 500$ outcomes.

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Elementary rules of counting: combining the rules together

- ▶ **Example:** How many odd three-digit numbers are there (allowing leading zeros, such as 007)?
- ▶ **Second solution:**
 - ▶ Number of outcomes = number of outcomes where the three-digit number ends in a 1 or 3 or 5 or 7 or 9.
 - ▶ Each of these five cases has $10 \times 10 = 100$ outcomes.
 - ▶ So there are $5 \times 100 = 500$ outcomes overall.

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Facts about playing cards

- ▶ A deck of cards contains 52 cards.
- ▶ Each card belongs to one of four *suits*
 - ♣ (Clubs), ♦ (Diamonds), ♥ (Hearts), ♠ (Spades)and one of thirteen denominations
 - 2, 3, 4, 5, 6, 7, 8, 9, 10, J(ack), Q(ueen), K(ing), A(ce).
- ▶ The clubs and spades are black and the diamonds and hearts are red.
- ▶ Unless otherwise specified, assume that for any example you begin with a complete deck and that as cards are dealt they are not immediately replaced back into the deck.
- ▶ We abbreviate a card using the denomination and then suit, such that $2♥$ (or 2H) represents the 2 of Hearts.

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Poker hands

- ▶ In standard poker you receive 5 cards.
- ▶ The suits are equally important.
- ▶ The face values are ordered
$$2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < J < Q < K < A$$
- ▶ While you can later discard cards and then replace them, for most of our examples we will only consider the initial configuration.
- ▶ *Pair (two of a kind)*: two cards that are the same denomination, such as a pair of 4's.
- ▶ *Three of a kind* and *four of a kind* are defined similarly.
- ▶ *Full house*: three of one kind and a pair of another kind.
- ▶ *Straight*: the cards are in sequential order, with no gaps.
- ▶ *Flush*: all five cards are of the same suit.
- ▶ *Straight flush*: all five cards are of the same suit and in sequential order (i.e., a straight *and* a flush).

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Poker hands (cont'd)

Ordering of the hands (highest to lowest):

- ▶ straight flush (with a "royal flush" [ace high] the highest possible hand of all)
- ▶ four of a kind
- ▶ full house
- ▶ flush
- ▶ straight
- ▶ three of a kind
- ▶ two pairs
- ▶ one pair
- ▶ high card

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A poker example

In how many ways can you draw a flush in poker, assuming that the order of the five cards drawn matters? (We will learn how to relax this assumption in the next section.)

- ▶ There are four basic ways to get a flush: all clubs or all diamonds or all hearts or all spades.
- ▶ Each is an outcome satisfying the condition of drawing a flush; we want to determine the total number of outcomes of these four *non-overlapping* outcomes.
- ▶ How many ways can we get an all-clubs flush? By multiplication rule to select 5 cards without replacement,

$$\# \text{ ways to draw five clubs} = 13 \times 12 \times 11 \times 10 \times 9 = 154,440.$$

- ▶ Therefore, by the addition rule, there are $4 \times 154,440 = 617,760$ ways to get a flush.

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Permutations and Combinations

- ▶ Sometimes order matters, sometimes it doesn't.
- ▶ **Example:** How many ways to get a royal flush in spades?

A♠, K♠, Q♠, J♠, 10♠

- ▶ If order matters, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.
- ▶ If order does not matter, there is only 1 way.
- ▶ Order matters: *permutation*
- ▶ Order doesn't matter: *combination*

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Permutations

- ▶ *Permutation*: order matters, cannot reuse objects.
- ▶ Phone numbers 123-456-7890 and 789-012-3456 are different. These are two permutations of the set of digits.
- ▶ **Example**: How many ways to seat 5 children in 5 chairs?
 - ▶ Both criteria for permutations are satisfied.
 - ▶ Counting permutations of 5 children.
 - ▶ By multiplication rule, there are

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

different seating arrangements.

- ▶ **Example**: How many ways to seat 10 children in 5 chairs?
 - ▶ Both criteria for permutations are satisfied.
 - ▶ Counting permutations of 10 children, *chosen 5 at a time*.
 - ▶ By multiplication rule, there are

$$10 \times 9 \times 8 \times 7 \times 6 = 30,240$$

different seating arrangements.

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Permutations (cont'd)

- ▶ **Notation**: $P(n, r)$ is the number of permutations of n objects, chosen r at a time.
- ▶ Formula for $P(n, r)$?

$$P(n, r) = n(n-1)(n-2)\dots(n-r+1)$$

- ▶ Excursus on factorials
 - ▶ $n!$ is the product of the natural numbers $1, 2, \dots, n$.
 - ▶ Semi-special case: $0! = 1$.
 - ▶ Table of factorials:

n	0	1	2	3	4	5	6	7	...
$n!$	1	1	2	6	24	120	720	5,040	...
n	...	8	9	10
$n!$...	40,320	362,880	3,628,800

- ▶ "Simplified" formula for $P(n, r)$?

$$P(n, r) = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

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Permutations (cont'd)

- ▶ **Example (cont'd)**: We have

$$P(10, 5) = 10 \times 9 \times 8 \times 7 \times 6 = 30,240.$$

- ▶ We also have

$$P(10, 5) = \frac{10!}{(10-5)!} = \frac{10!}{5!}.$$

- ▶ Save some work: cancel common factors

$$\begin{aligned} P(10, 5) &= \frac{10!}{(10-5)!} = \frac{10!}{5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\ &= 10 \times 9 \times 8 \times 7 \times 6 = 30,240. \end{aligned}$$

- ▶ All our answers agree.

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Permutations (cont'd)

- ▶ Sanity check:
 - ▶ $P(n, r)$ counts something.
 - ▶ Thus $P(n, r)$ must be a non-negative integer.
 - ▶ The formula


$$P(n, r) = \frac{n!}{(n-r)!}$$

appears to involve division.

- ▶ You will *always* be able to use the cancellation trick to get rid of divisions.
- ▶ Alternatively, use the formula

$$P(n, r) = n(n-1)(n-2)\dots(n-r+1).$$

(There are r factors.)

- ▶  If the answer you get to a permutation problem is anything other than a non-negative integer, *go back and check your work!*

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Permutations (cont'd)

- ▶ **Example:** In how many ways can we choose a 3-person slate of officers (president, vice-president, secretary) out of the 10 members in this class?
- ▶ **Solution:** We need to choose 3 distinct people out of 10, with order mattering.
- ▶ So

$$P(10, 3) = 10 \times 9 \times 8 = 720.$$

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Permutations (cont'd)

- ▶ **Example:** In major league baseball, each team has a 25-player roster. How many possible batting orders are there for such a roster?
- ▶ **Solution:** Check that this is a permutation.
- ▶ Total number of batting orders is

$$P(25, 25) = \frac{25!}{0!} = 25 \times 24 \times \dots \times 17 = 741,354,768,000.$$

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Combinations

- ▶ For some problems, order matters. (Baseball lineup problem.)
- ▶ For some problems, order does *not* matter.
- ▶ **Example:** We need to choose a 12-person jury from a pool of 1000 people. The order does not matter here. We want the number of *combinations* of 1000 persons, chosen 12 at a time.
- ▶ **Notation:** $C(n, r)$ denotes the number of *combinations* of n objects, chosen r at a time. Here the order *does not* matter, and we are not allowed to reuse objects. We often read this as " n choose r ".
- ▶ Formula for combinations:

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Why?

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!r!}$$

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Combinations (cont'd)

- ▶ **Example:** In how many ways can we choose a 3-person committee out of a 10-member class?
- ▶ **Solution:** We need to choose 3 distinct people out of 10, with order not mattering.
- ▶ So

$$\begin{aligned} C(10, 3) &= \frac{10!}{3! \cdot 7!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\ &= \frac{10 \times \overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}} \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(\cancel{3} \times \cancel{2} \times \cancel{1})(\cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1})} \\ &= 10 \times 3 \times 4 = 120. \end{aligned}$$

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Combinations (cont'd)

► Sanity check:

- $C(n, r)$ counts something.
- Thus $C(n, r)$ must be a non-negative integer.
- The formula

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

involves division.

- You will *always* be able to use the cancellation trick to get rid of divisions.



- If the answer you get to a combination problem is anything other than a non-negative integer, *go back and check your work!*

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Additional Examples

Example: A typical telephone number has 10 digits (e.g., 555-817-4495), where the first three are known as the area code and the next three as the exchange.

1. Assuming *no* restrictions, how many possible (three-digit) area codes are there?
Solution: $10 \times 10 \times 10 = 1,000$ three-digit area codes.
2. Assuming that the middle digit of the area code must be a 0 or a 1 (which was required until recently), how many possible (3 digits) area codes are there?
Solution: $10 \times 2 \times 10 = 200$ area codes.
3. Assuming no restrictions whatsoever, how many possible values are there for the full 10-digit phone number?
Solution: $10^{10} = 10,000,000,000$ phone numbers
4. If the only restriction is that no digit may be used more than once, how many possible 10-digit phone numbers are there?
Solution: $10 \times 9 \times \dots \times 1 = 10! = 3,628,800$ phone numbers.

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Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt “two pairs”?

Solution:

- There are $C(13, 2)$ ways to identify the two denominations.
- For each denomination, there are $C(4, 2)$ ways to choose two of the four cards. Do this twice.
- Pick the last card? 11 ways for each of 4 suits.
- Final answer:

$$C(13, 2) \times C(4, 2) \times C(4, 2) \times 11 \times 4 = 123,552 \text{ ways.}$$

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Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt “three of a kind”?

Solution:

- We can choose the denomination with 3 of a kind in 13 ways.
- There are $C(4, 3)$ ways to choose the three cards of said denomination.
- The two remaining cards must come from the other 12 denominations. They can't be the same, since this would yield a full house. Since there are 4 suits, there are $C(12, 2) \times 4 \times 4$ ways of choosing these two cards.
- Final answer:

$$13 \times C(4, 3) \times C(12, 2) \times 4 \times 4 = 54,912 \text{ ways.}$$

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Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt a “full house”?

Solution:

- ▶ A full house requires 3 of a kind and also 2 of a different kind.
- ▶ We can choose the denomination with 3 of a kind in 13 ways and then we can choose the 3 specific cards in $C(4,3)$ ways.
- ▶ Then we can choose the denomination with the 2 of a kind in 12 ways and choose the 2 specific cards in $C(4,2)$ ways.
- ▶ Final answer:

$$13 \times C(4,3) \times 12 \times C(4,2) = 3,744 \text{ ways.}$$

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Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word MISSISSIPPI?

Solution #1:

- ▶ Multiplication rule: $11!$ ways.
- ▶ Since there are
 - ▶ 4 instances of S and I
 - ▶ 2 instances of P
 not all $11!$ ways are distinguishable.
- ▶ Since there are 4 instances of S, their appearance can be permuted in $4!$ different ways. So we need to divide the current answer by $4!$, getting $11!/4!$.
- ▶ Since there are 4 instances of I, their appearance can be permuted in $4!$ different ways. So we need to divide the current answer by $4!$, getting $11!/(4!4!)$.

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Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word MISSISSIPPI?

Solution #1 (contd):

- ▶ Answer so far (accounting for multiple S and I): $11!/(4!4!)$.
- ▶ Since there are 2 instances of P, their appearance can be permuted in $2!$ different ways. So we need to divide the current answer by $2!$, getting $11!/(4!4!2!)$.
- ▶ Final answer:

$$\frac{11!}{4!4!2!} = 11 \times 10 \times 9 \times 7 \times 5 = 34,650 \text{ ways.}$$

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Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word MISSISSIPPI?

Solution #2: Use a “fill-in-the-blank” approach, starting with 11 blanks

— — — — — — — — — — — — — — —

- ▶ Can assign the one M in $C(11,1) = 11!/(10! \times 1!) = 11$ ways.
- ▶ Can assign the two P's in $C(10,2)$ ways.
- ▶ Can assign the four S's in $C(8,4)$ ways.
- ▶ Can assign the four I's in $C(4,4) = 1$ way.
- ▶ Total number of ways is then

$$C(11,1) \times C(10,2) \times C(8,4) \times C(4,4) = 11 \times 45 \times 70 \times 1 = 34,650.$$

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