CISC 1100/1400
Structures of Comp. Sci./Discrete Structures
Chapter 6
Counting

Arthur G. Werschulz

Fordham University Department of Computer and Information Sciences

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  - Methodically enumerating a set.
- Connection between counting and probability theory.

Outline

- Counting and how to count
- Elementary rules for counting
  - The addition rule
  - The multiplication rule
  - Using the elementary rules for counting together
- Permutations and combinations
- Additional examples

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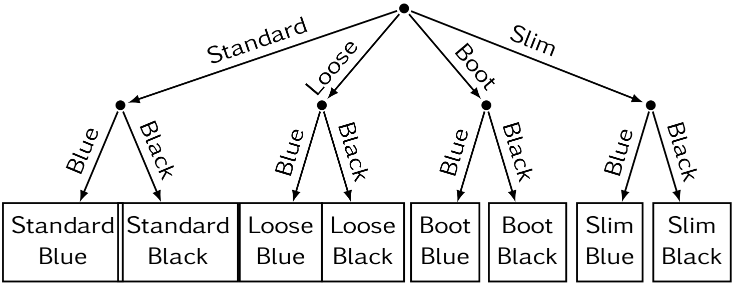
  - What if more than two "features"?
- One idea: Use a *tree structure* to help you enumerate the choices.



Counting and how to count (cont'd)

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- How many configurations? 8.
- How to count configurations without listing?

Elementary rules of counting

- Two basic rules:
  - Addition rule
  - Multiplication rule
- Using these rules together

- **Example:** You need to purchase one shirt of any kind. The store has five short sleeve shirts and eight long sleeve shirts. How many possible ways are there to choose a shirt?

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  - If we have k choices $C_1, \dots, C_k$ having $n_1, \dots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *or* C_2 occurring *or* … *or* C_k occurring is $n_1 + n_2 + \cdots + n_k$.

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- Fairly straightforward.

- In our jeans example,

$$\text{\# of jeans configurations} = (\text{\# number of styles}) \times (\text{\# of colors})$$

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  - More generally, if we have k choices $C_1, \ldots, C_k$ having $n_1, \ldots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *and* C_2 occurring *and* ... *and* C_k occurring is $n_1 \times n_2 \times \cdots \times n_k$.

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  - More generally, if we have k choices $C_1, \ldots, C_k$ having $n_1, \ldots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *and* C_2 occurring *and* ... *and* C_k occurring is $n_1 \times n_2 \times \cdots \times n_k$.
- Roughly speaking:
  - addition rule: "or" rule
  - multiplication rule: "and" rule

Elementary rules of counting: the multiplication rule (cont'd)

Example: Solve jeans problem via multiplication rule . . .

- four styles (standard, loose, slim, and boot fits) and
- two colors (black, blue)

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Solution: Our choices?

$$C_1 = \text{“choose the jeans style”},$$
$$C_2 = \text{“choose the jeans color”}.$$

Our outcomes?

$$O_1 = \{\text{standard fit, loose fit, boot fit, slim fit}\},$$
$$O_2 = \{\text{black, blue}\}.$$

Now determine the cardinalities of the sets:

$$n_1 = |O_1| = 4 \qquad n_2 = |C_2| = 2.$$

Now we apply the multiplication rule

$$\text{Total number of outcomes} = n_1 \times n_2 = 4 \times 2 = 8.$$

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- The set of possible outcomes is for O_1 *and* O_2 occurring is $O_1 \times O_2$.
- We know that $|O_1 \times O_2| = |O_1| \cdot |O_2|$.
- This is the multiplication rule!

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Example: Suppose that you flip a coin twice and record the outcome (head or tail) for each flip. How many possible outcomes are there?

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Solution: There are two choices, C_1 and C_2, corresponding to the two coin flips. C_1 *and* C_2 must occur, so the multiplication rule applies. Each choice has two possible outcomes, thus $n_1 = 2$ and $n_2 = 2$. Thus by the multiplication principle of counting, there are $2 \times 2 = 4$ possible outcomes.

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Example: You are asked to flip a coin five times and to record the outcome (head or tail) for each flip. How many possible outcomes are there?

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- This example differs from the previous one only in that there are five choices instead of two.
- For each choice there are two possible outcomes.
- The total number of outcomes is

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. If a number may be selected more than once, then how many ways can you fill out the lottery card?

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- Each choice has twenty possible outcomes (i.e., you pick a number between 1 and 20).
- There are

$$20 \times 20 \times 20 \times 20 \times 20 = 20^5 = 3{,}200{,}000$$

possible ways to fill out the lottery card.

Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. The numbers are chosen by the lottery commission from a bin and once a number is chosen it is discarded and cannot be chosen again. In how many ways can you fill out the lottery card?

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- The number of outcomes for C_1 is 20, for C_2 is 19, for C_3 is 18, for C_4 is 17 and for C_5 is 16.

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- Thus the number of possible outcomes is

$$20 \times 19 \times 18 \times 17 \times 16 = 1{,}860{,}480.$$

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- **Solution:** $5 + 2 = 7$ ways.

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  - Hence there are $10 \times 10 \times 5 = 500$ outcomes.

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  - So there are $5 \times 100 = 500$ outcomes overall.

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  and one of thirteen denominations

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- Unless otherwise specified, assume that for any example you begin with a complete deck and that as cards are dealt they are not immediately replaced back into the deck.
- We abbreviate a card using the denomination and then suit, such that 2$\heartsuit$ (or 2H) represents the 2 of Hearts.

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Poker hands (cont'd)

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- straight flush (with a "royal flush" [ace high] the highest possible hand of all)
- four of a kind
- full house
- flush
- straight
- three of a kind
- two pairs
- one pair
- high card

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In how many ways can you draw a flush in poker, assuming that the order of the five cards drawn matters? (We will learn how to relax this assumption in the next section.)

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- Therefore, by the addition rule, there are $4 \times 154{,}440 = 617{,}760$ ways to get a flush.

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    (There are r factors.)
- If the answer you get to a permutation problem is anything other than a non-negative integer, *go back and check your work*!

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- Total number of batting orders is

$$P(25,9) = \frac{25!}{16!} = 25 \times 24 \times \cdots \times 17 = 741{,}354{,}768{,}000.$$

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- If the answer you get to a combination problem is anything other than a non-negative integer, *go back and check your work*!

Additional Examples

Example: A typical telephone number has 10 digits (e.g., 555-817-4495), where the first three are known as the area code and the next three as the exchange.

1. Assuming *no* restrictions, how many possible (three-digit) area codes are there?

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- Pick the last card? 11 ways for each of 4 suits.
- Final answer:

$$C(13,2) \times C(4,2) \times C(4,2) \times 11 \times 4 = 123{,}552 \text{ ways.}$$

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- There are $C(4,3)$ ways to choose the three cards of said denomination.
- The two remaining cards must come from the other 12 denominations. They can't be the same, since this would yield a full house. Since there are 4 suits, there are $C(12,2) \times 4 \times 4$ ways of choosing these two cards.

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- Final answer:

$$13 \times C(4,3) \times C(12,2) \times 4 \times 4 = 54{,}912 \text{ ways.}$$

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- Final answer:

$$13 \times C(4,3) \times 12 \times C(4,2) = 3{,}744 \text{ ways.}$$

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  not all $11!$ ways are distinguishable.
- Since there are 4 instances of S, their appearance can be permuted in 4! different ways. So we need to divide the current answer by 4!, getting $11!/4!$.

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- Since there are 4 instances of I, their appearance can be permuted in 4! different ways. So we need to divide the current answer by 4!, getting $11!/(4!4!)$.

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Solution #1 (contd):

- Answer so far (accounting for multiple `S` and `I`): $11!/(4!4!)$.

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- Answer so far (accounting for multiple S and I): $11!/(4!4!)$.
- Since there are 2 instances of P, their appearance can be permuted in $2!$ different ways. So we need to divide the current answer by $2!$, getting $11!/(4!4!2!)$.

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- Since there are 2 instances of `P`, their appearance can be permuted in $2!$ different ways. So we need to divide the current answer by $2!$, getting $11!/(4!4!2!)$.
- Final answer:

$$\frac{11!}{4!4!2!} = 11 \times 10 \times 9 \times 7 \times 5 = 34{,}650 \text{ ways.}$$

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How many distinguishable ways are there to arrange the letters in the word `MISSISSIPPI`?

Solution #2: Use a "fill-in-the-blank" approach, starting with 11 blanks

____ ____ ____ ____ ____ ____ ____ ____ ____ ____ ____

- Can assign the one `M` in $C(11, 1) = 11!/(10! \times 1!) = 11$ ways.

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- Can assign the four I's in $C(4,4) = 1$ way.
- Total number of ways is then

$$C(11,1) \times C(10,2) \times C(8,4) \times C(4,4) = 11 \times 45 \times 70 \times 1 = 34{,}650.$$