CISC 1100/1400
Structures of Comp. Sci./Discrete Structures
Chapter 7
Probability

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Summer, 2017

Why study probability?

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Historical note: A gambler's dispute in 1654 led Blaise Pascal and Pierre de Fermat to create a mathematical theory of probability.

Outline

- Terminology and background
- Complement
- Elementary rules for probability
- General rules for probability
- Bernoulli trials and probability distributions
- Expected value

Terminology and background

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- *Event*: set E of desired outcomes
- *Sample space*: (finite) set S of all possible outcomes
- The *probability* $\text{Prob}(E)$ of an event $E \subseteq S$ is given as

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

Terminology and background (cont'd)

Example: What is the probability of getting 7 when rolling two dice?

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- Our sample space S is

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S = \{&(1,1),(1,2),\ldots,(1,6),\\
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&(3,1),(3,2),\ldots,(3,6),\\
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- So

$$\text{Prob}(E) = \frac{|E|}{|S|} = \tfrac{6}{36} = \tfrac{1}{6} = 0.1667.$$

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Example: What is the probability of getting 7 when rolling two dice (cont'd)?

Discussion:

- Since E and S are small, could solve by enumeration.

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- If either E or S is big, this is impractical. (Poker problems, lottery problems, ...).
- Can often use counting principles from previous chapter to determine $|S|$ and/or $|E|$.
- In our case, there are 6 outcomes for the roll of each of the two dice.
- So multiplication principle tells us that there are $6 \times 6 = 36$ outcomes for the roll of both dice, i.e., $|S| = 36$.

Terminology and background (cont'd)

- Since $0 \leq |E| \leq |S|$ and $\text{Prob}(E) = |E|/|S|$, we have
$$0 \leq \text{Prob}(E) \leq 1$$
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- **Example:** For rolling two dice

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$$\text{Prob(roll value is negative)} = 0$$

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- **Example:** For rolling two dice

$$\text{Prob}(\text{roll value is positive}) = 1$$
$$\text{Prob}(\text{roll value is negative}) = 0$$

- If you ever calculate a probability as being negative or being greater than 1, you've made a mistake.

Terminology and background (cont'd)

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  - Example: Throwing a loaded die.
  - Example: Choosing a ball out of a bag, where some balls are larger than others.

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Terminology and background (cont'd)

Determine the probability that you flip a coin 10 times and that you get all heads.

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Determine the probability that you flip a coin 10 times and that you get all heads.

Solution:

- S is too big to enumerate.
- Use the multiplication rule:

$$|S| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10} = 1{,}024$$

$$|E| = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1^{10} = 1$$

$$\text{Prob}(E) = \frac{|E|}{|S|} = \tfrac{1}{1{,}024} = 0.00098.$$

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- Of course, we really know $|E| = 1$ directly, since

$$E = \{(\mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H}, \mathrm{H})\}$$

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E &= \{(\mathrm{H,H,T}), (\mathrm{H,T,H}), (\mathrm{T,H,H})\} \\
\mathrm{Prob}(E) &= \frac{|E|}{|S|} = \tfrac{3}{8} = 0.375.
\end{aligned}
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Terminology and background (cont'd)

Determine the probability that you flip a coin ten times and that you get exactly two heads.

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This is an example of a Bernoulli trial. We'll look at this later.

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$$\begin{aligned}|S| &= 52,\\ E &= \{2\clubsuit, 2\diamondsuit, 2\heartsuit, 2\spadesuit, 3\clubsuit, 3\diamondsuit, 3\heartsuit, 3\spadesuit\}, \text{ so that } |E| = 8,\\ \text{Prob}(E) &= \tfrac{8}{52} = \tfrac{2}{13} = 0.154.\end{aligned}$$

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- Probability of a complement?
$$\text{Prob}(E') = 1 - \text{Prob}(E).$$
- Why? Since
$$|E'| = |S| - |E|,$$
it follows that
$$\text{Prob}(E') = \frac{|E'|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - \text{Prob}(E).$$

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  - computing $\text{Prob}(E) = 1 - \text{Prob}(E')$.

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- Easy to see that $|S| = 12^6 = 2{,}985{,}984$.

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- Easy to see that $|S| = 12^6 = 2{,}985{,}984$.
- Calculating $|E|$: seems hard, so calculate $|E'|$ instead:

$$|E'| = P(12,6) = 12 \times 11 \times 10 \times 9 \times 8 \times 7 = 665{,}280.$$

Thus

$$\text{Prob}(E') = \frac{|E'|}{|S|} = \frac{665{,}280}{2{,}985{,}984} = 0.2228 = 22.28\%$$

So

$$\text{Prob}(E) = 1 - \text{Prob}(E') = 1 - 0.2228 = 0.7772 = 77.72\%$$

Elementary rules for probability

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- To properly state these rules, we need two new concepts: *disjointness* and *independence*:
  - Two or more events are *disjoint* if the outcomes associated with one event are not present in the outcomes of any of the other events (i.e., if the events form non-overlapping sets).
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    More formally, two events E_1 and E_2 are *disjoint* if $E_1 \cap E_2 = \emptyset$.
  - Two events E_1 and E_2 are *independent* if the outcome of any one of these events does not *in any way* impact or influence the outcome of the other event.
    More formally, two events E_1 and E_2 are *independent* if

$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2).$$

Elementary rules for probability (cont'd)

- **Example:** A six-sided die is rolled. Are the events "roll an odd number" and "roll an even number" disjoint?

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- **Solution:** Let

$$E_1 = \text{“roll an odd number”} = \{1, 3, 5\}$$

and

$$E_2 = \text{“roll an even number”} = \{2, 4, 6\}.$$

Since $E_1 \cap E_2 = \emptyset$ contains no elements, the events are disjoint.

Elementary rules for probability (cont'd)

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Second part establishes a useful general result: *disjoint events (having non-zero probabilities) are never independent.*

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- **Example:** Two cards are drawn from a fresh deck without replacement. Are the outcomes associated with each draw disjoint?

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- **Solution:** Let E_1 and E_2 denote the events of drawing the first and second cards from the deck. Clearly $E_1 \cap E_2$ consists of 51 possibilities. Thus $E_1 \cap E_2 \neq \emptyset$, and so the outcomes are *not* disjoint.

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  - For example, if A♠ is drawn on the first draw then it cannot be drawn on the second draw.

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  - So $\text{Prob}(E_1) \cdot \text{Prob}(E_2) = \frac{1}{52} \cdot \frac{1}{52} = \frac{1}{2{,}704}$.

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  - Since $\text{Prob}(E_1 \cap E_2) \neq \text{Prob}(E_1) \cdot \text{Prob}(E_2)$, the events are not independent.

Addition rule for probability

- Let E_1 and E_2 be disjoint events. Then

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  - So

$$\begin{aligned}\text{Prob}(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} \\ &= \text{Prob}(E_1) + \text{Prob}(E_2).\end{aligned}$$

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- We have

$$\begin{aligned}\text{Prob(ace, then value=10)} &= \text{Prob(ace on first draw)} \\ &\quad \times \text{Prob(value=10 on second draw)} \\ &= \tfrac{4}{52} \times \tfrac{16}{51} = 0.024 \\ \text{Prob(value=10, then ace)} &= \text{Prob(value=10 on first draw)} \\ &\quad \times \text{Prob(ace on second draw)} \\ &= \tfrac{16}{52} \times \tfrac{4}{51} = 0.024.\end{aligned}$$

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- Thus $\text{Prob(blackjack)} = 0.024 + 0.024 = 0.048$ (or 4.8%).

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- **Solution:** The events $E_1 = \{1\}$ and $E_2 = \{2\}$ are disjoint. So

$$\begin{aligned}\text{Prob}(1 \text{ or } 2) &= \text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) \\ &= \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3} = 0.333.\end{aligned}$$

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- This was easy enough to do directly, since we're interested in $\text{Prob}(E)$ for $E = \{1, 2\}$. So

$$\text{Prob}(E) = \frac{|E|}{|S|} = \tfrac{2}{6} = \tfrac{1}{3} = 0.333.$$

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- **Why?** This is simply the formal definition of independence.

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- Constituent events

$$E_i = \text{“toss \# } i \text{ is a head”} \qquad (i = 1, 2, 3)$$

are intuitively independent, and so

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- Clearly

$$\text{Prob}(E_1) = \text{Prob}(E_2) = \text{Prob}(E_3) = \tfrac{1}{2}.$$

- So

$$\text{Prob}(E) = \tfrac{1}{8} = 0.125.$$

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  Can't decide, so choose randomly. What is the probability that you will wind up with a red car, with air-conditioning, but without the 4-wheel drive option?

Multiplication rule for probability (cont'd)

- **Solution:** Let E denote the event of interest (red car, A/C, no 4WD). Constituent events are

$$E_1 = \text{“choose red color”},$$
$$E_2 = \text{“choose A/C option”},$$
$$E_3 = \text{“don't choose 4WD option”}.$$

with corresponding sample spaces of sizes

$$|S_1| = 8, |S_2| = 2, |S_3| = 2.$$

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- Since $|E_1| = |E_2| = |E_3| = 1$, we have

$$\text{Prob}(E_1) = \tfrac{1}{8}, \text{Prob}(E_2) = \tfrac{1}{2}, \text{Prob}(E_3) = \tfrac{1}{2}.$$

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- Since the events are intuitively independent, we have

$$\begin{aligned}\text{Prob}(E) &= \text{Prob}(E_1 \cap E_2 \cap E_3) \\ &= \text{Prob}(E_1) \cdot \text{Prob}(E_2) \cdot \text{Prob}(E_3) \\ &= \tfrac{1}{8} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{32} = 0.031.\end{aligned}$$

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  - Note that $|E_1 \cup E_2| \neq |E_1| + |E_2| = 30$. Since there are two black 2's and two red 2's, we don't want to double-count the red 2's! So $|E_1 \cup E_2| = 28$.

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  - So
  $$\text{Prob}(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \tfrac{28}{52} = 0.538.$$

General addition rule for probability

- For *any* two events E_1 and E_2,

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2)$$

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- **Why?** Simple consequence of the inclusion/exclusion rule

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

General addition rule for probability

- For *any* two events E_1 and E_2,

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2)$$

- **Why?** Simple consequence of the inclusion/exclusion rule

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

- Note that this reduces to the earlier rule

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) \qquad \text{for disjoint } E_1, E_2$$

General addition rule for probability (cont'd)

- **Example:** Given a standard deck of cards, what is the probability of drawing a red card *or* a 2?

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- **Solution:**
  - $|S| = 52$, as before.
  - Let $E_1 =$ "pick a red card". Then $|E_1| = 26$, and so $\text{Prob}(E_1) = \frac{26}{52} = \frac{1}{2}$.

General addition rule for probability (cont'd)

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  - Let $E_2 =$ "pick a 2". Then $|E_2| = 4$, and so $\text{Prob}(E_2) = \frac{4}{52} = \frac{1}{13}$.

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  - Since there are two red 2's, $|E_1 \cap E_2| = 2$, and so $\text{Prob}(E_1 \cap E_2) = \frac{2}{52} = \frac{1}{26}$.

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  - Since there are two red 2's, $|E_1 \cap E_2| = 2$, and so $\text{Prob}(E_1 \cap E_2) = \frac{2}{52} = \frac{1}{26}$.
  - Thus

$$\begin{aligned}\text{Prob}(E_1 \cup E_2) &= \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2) \\ &= \frac{1}{2} + \frac{1}{13} - \frac{1}{26} = \frac{7}{13} \\ &= 0.538\end{aligned}$$

- **Example:** I flip a coin and roll a six-sided die. What is the probability that I get a head or roll either a 1 or a 2 ?

General addition rule for probability (cont'd)

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- **Solution:**
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$$
\begin{aligned}
S_1 &= \{\text{head}, \text{tails}\}, \text{ and so } |S_1| = 2, \\
S_2 &= \{1, 2, 3, 4, 5, 6\}, \text{ and so } |S_2| = 6, \\
E_1 &= \text{“flip coin and get head”}, \text{ and so } |E_1| = 1, \\
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  - Thus

$$\begin{aligned} \text{Prob}(E_1) &= \frac{|E_1|}{|S_1|} = \tfrac{1}{2} \\ \text{Prob}(E_2) &= \frac{|E_2|}{|S_2|} = \tfrac{2}{6} = \tfrac{1}{3}. \end{aligned}$$

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$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2) = \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6}.$$

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  - Hence

$$\begin{aligned}\text{Prob}(E_1 \cup E_2) &= \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2) \\ &= \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{6} = \tfrac{2}{3} = 0.667.\end{aligned}$$

General multiplication rule for probability

- Recall that

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- If E_1, E_2 are independent, then $\text{Prob}(E_1|E_2) = \text{Prob}(E_1)$.

General multiplication rule for probability (cont'd)

- **Example:** A standard six-sided die is rolled by your friend, but in such a way that you cannot see what value comes up. Your friend tells you that the value that comes up is odd, but does not tell you what specific value was rolled. What is the probability that she rolled a 3?

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$$\text{Prob}(E_1) = \frac{1}{6}.$$

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  - Since the roll was odd, $E_2 = \{1, 3, 5\}$.

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- **Solution:**
  - Let $E_1 =$ "a 3 is rolled" and $E_2 =$ "the roll is odd". Then

$$\text{Prob}(E_1) = \tfrac{1}{6}.$$

  - Since the roll was odd, $E_2 = \{1, 3, 5\}$.
  - Since $|E_2| = 3$ and one of the three elements of E_2 is 3,

$$\text{Prob}(E_1|E_2) = \tfrac{1}{3}.$$

General multiplication rule for probability (cont'd)

- Rule for computing conditional probability:

$$\text{Prob}(E_1|E_2) = \frac{\text{Prob}(E_1 \cap E_2)}{\text{Prob}(E_2)}.$$

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  - Since the event E_2 has happened, think of E_2 as a new sample space.

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  - Since the event E_2 has happened, think of E_2 as a new sample space.
  - $\text{Prob}(E_1|E_2)$ may be interpreted as the fraction of E_2 that is covered by E_1 elements.
  - But these E_1 elements must really be elements of $E_1 \cap E_2$.

General multiplication rule for probability (cont'd)

- The general multiplication rule is now given by

$$\begin{aligned}\text{Prob}(E_1 \cap E_2) &= \text{Prob}(E_2) \cdot \text{Prob}(E_1|E_2) \\ &= \text{Prob}(E_1) \cdot \text{Prob}(E_2|E_1)\end{aligned}$$

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  - First line follows from definition

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- **Why?**
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  - Second line:

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General multiplication rule for probability (cont'd)

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  - Let $E_1 =$ "pick Ace on first draw" and $E_2 =$ "pick Ace on second draw".

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- **Example:** What is the probability of picking 2 aces from a deck of cards?
- **Solution:**
  - Let $E_1 =$ "pick Ace on first draw" and $E_2 =$ "pick Ace on second draw".
  - $\text{Prob}(E_1) = \frac{4}{52}$.

General multiplication rule for probability (cont'd)

- **Example:** What is the probability of picking 2 aces from a deck of cards?
- **Solution:**
  - Let $E_1 =$ "pick Ace on first draw" and $E_2 =$ "pick Ace on second draw".
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  - E_1 and E_2 are not independent: drawing the first card changes the deck.

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  - $\text{Prob}(E_2|E_1) = \frac{3}{51}$.

General multiplication rule for probability (cont'd)

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  - E_1 and E_2 are not independent: drawing the first card changes the deck.
  - $\text{Prob}(E_2|E_1) = \frac{3}{51}$.
  - So
$$\begin{aligned}\text{Prob}(E_1 \cap E_2) &= \text{Prob}(E_1) \cdot \text{Prob}(E_2|E_1) \\ &= \frac{4}{52} \times \frac{3}{51} = 0.0045.\end{aligned}$$

Bernoulli trials and probability distributions

- What is the probability of flipping a coin ten times and getting three heads?

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- Heart of such problems? We′re repeating an experiment having two outcomes, say O_1 and O_2.
  - Let $p = \text{Prob}(O_1)$.
  - Then $\text{Prob}(O_2) = 1 - p$.
- Out of n repetitions, we want to know the probability that the first outcome happens k times.

- We have n slots to fill, with k instances of O_1 and $n - k$ of O_2.

Bernoulli trials and probability distributions (cont'd)

- We have n slots to fill, with k instances of O_1 and $n-k$ of O_2.
- We label each O_1-slot by p, and each O_2-slot by $1-p$.

Bernoulli trials and probability distributions (cont'd)

- We have n slots to fill, with k instances of O_1 and $n - k$ of O_2.
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- The probability of any given choice is $p^k(1 - p)^{n-k}$.

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$$\text{Prob}(O_1 \text{ happens } k \text{ times }) = C(n,k)p^k(1-p)^{n-k}.$$

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  - Variance (a measure of spread) is $np(1-p)$.

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  - So the probability is given by

$$C(10, 2) \cdot (\tfrac{1}{2})^2 \cdot (\tfrac{1}{2})^8 = 45 \cdot (\tfrac{1}{2})^{10} = \tfrac{45}{1024} = 0.0439.$$

Bernoulli trials and probability distributions (cont′d)

- **Example:** What is the probability of rolling a pair of dice ten times and getting three 7′s?

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Bernoulli trials and probability distributions (cont'd)

- **Example:** What is the probability of rolling a pair of dice ten times and getting three 7's?
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  - This is a Bernoulli trial with

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Bernoulli trials and probability distributions (cont'd)

- **Example:** What is the probability of rolling a pair of dice ten times and getting three 7's?
- **Solution:**
  - This is a Bernoulli trial with

$$p = \frac{1}{6}, n = 10, k = 3.$$

  - So the probability is given by

$$C(10,3) \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^7 = 120 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^7 = \frac{390{,}625}{2{,}519{,}424} = 0.1550.$$

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- Let E be an event with outcomes $O_1, O_2, \ldots O_n$. Then

$$\text{Expected value of } E = \sum_{j=1}^{n} O_j \cdot \text{Prob}(O_j) = O_1 \cdot \text{Prob}(O_1) + O_2 \cdot \text{Prob}(O_2) + \cdots + O_n \cdot \text{Prob}(O_n).$$

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- Takes into account the fact that different outcomes may have different probabilities.

Expected Value (cont'd)

- **Example:** What's the expected value when one tosses a fair six-sided die once?

Expected Value (cont'd)

- **Example:** What's the expected value when one tosses a fair six-sided die once?
- **Solution:** The outcomes are

$$O_1 = 1, O_2 = 2, \ldots, O_6 = 6.$$

Expected Value (cont'd)

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- So

$$\begin{aligned} \text{Expected value} &= \tfrac{1}{6} \times 1 + \tfrac{1}{6} \times 2 + \tfrac{1}{6} \times 3 + \tfrac{1}{6} \times 4 + \tfrac{1}{6} \times 5 + \tfrac{1}{6} \times 6 \\ &= \tfrac{1}{6} + \tfrac{2}{6} + \tfrac{3}{6} + \tfrac{4}{6} + \tfrac{5}{6} + \tfrac{6}{6} \\ &= \tfrac{21}{6} = 3.5 \end{aligned}$$

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- Why? Since the n events are equally likely, they each have probability $1/n$.

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  - Further simplification: only one winning ticket.

Expected Value (cont'd)

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Expected Value (cont’d)

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  - The total number of ways 6 balls can be chosen out of 59 is

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